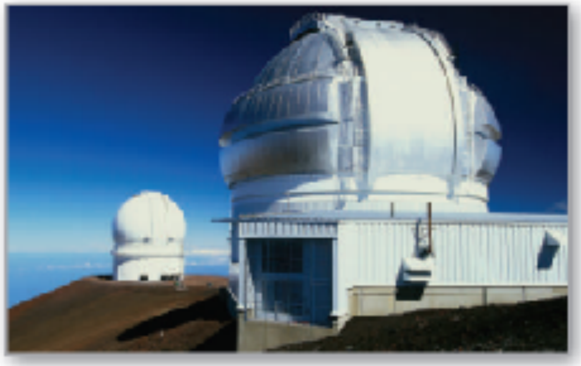
P Preparation for Calculus







P.1 Graphs and Models

Objectives

- Sketch the graph of an equation.
- Find the intercepts of a graph.
- Test a graph for symmetry with respect to an axis and the origin.
- Find the points of intersection of two graphs.
- Interpret mathematical models for real-life data.

The Graph of an Equation

The Graph of an Equation

Consider the equation $3x + y = 7$. The point (2, 1) is a **solution point** of the equation because the equation is satisfied (is true) when 2 is substituted for x and 1 is substituted for y. This equation has many other solutions, such as (1, 4) and (0, 7).

To find other solutions systematically, solve the original equation for y.

$y = 7 - 3x$ Analytic approach

The Graph of an Equation

Then construct a **table of values** by substituting several values of *x.*

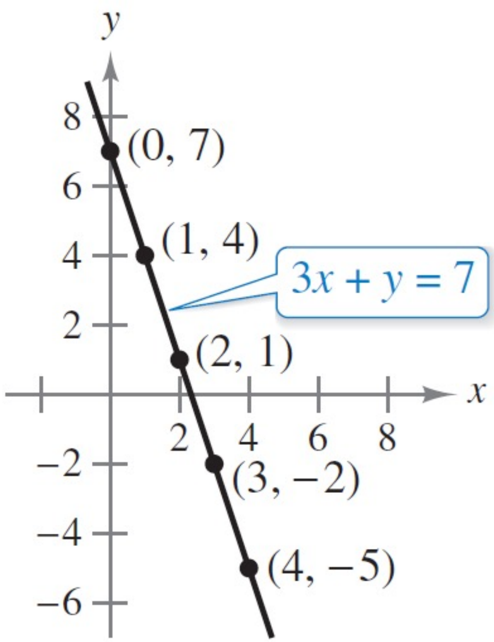
x	0	1	2	3	4
y	7	4	1	-2	-5

Numerical approach

From the table, you can see that (0, 7), (1, 4), (2, 1), (3, –2), and (4, –5) are solutions of the original equation $3x + y = 7$.

The Graph of an Equation

Like many equations, this equation has an infinite number of solutions. The set of all solution points is the **graph** of the equation, as shown in Figure P.1.



Graphical approach: $3x + y = 7$

Figure P.1

Example 1 – *Sketching a Graph by Point Plotting*

Sketch the graph of $y = x^2 - 2$.

Solution:

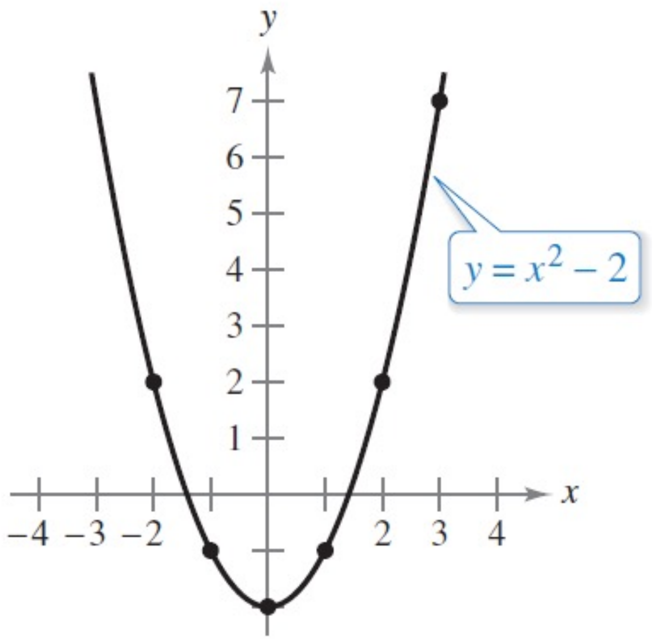
First construct a table of values. Next, plot the points shown in the table.

x	-2	-1	0	1	2	3
y	2	-1	-2	-1	2	7

Example 1 – *Solution*

cont'd

Then connect the points with a smooth curve, as shown in Figure P.2. This graph is a **parabola**.



The parabola $y = x^2 - 2$

Figure P.2

Intercepts of a Graph

Intercepts of a Graph

Two types of solution points that are especially useful in graphing an equation are those having zero as their *x*- or *y*-coordinate.

Such points are called **intercepts** because they are the points at which the graph intersects the *x*- or *y*-axis.

The point (a, 0) is an ***x*-intercept** of the graph of an equation if it is a solution point of the equation.

Intercepts of a Graph

To find the x-intercepts of a graph, let y be zero and solve the equation for x.

The point $(0, b)$ is a ***y*-intercept** of the graph of an equation when it is a solution point of the equation.

To find the y-intercepts of a graph, let x be zero and solve the equation for y.

Intercepts of a Graph

It is possible for a graph to have no intercepts, or it might have several. For instance, consider the four graphs shown in Figure P.5.

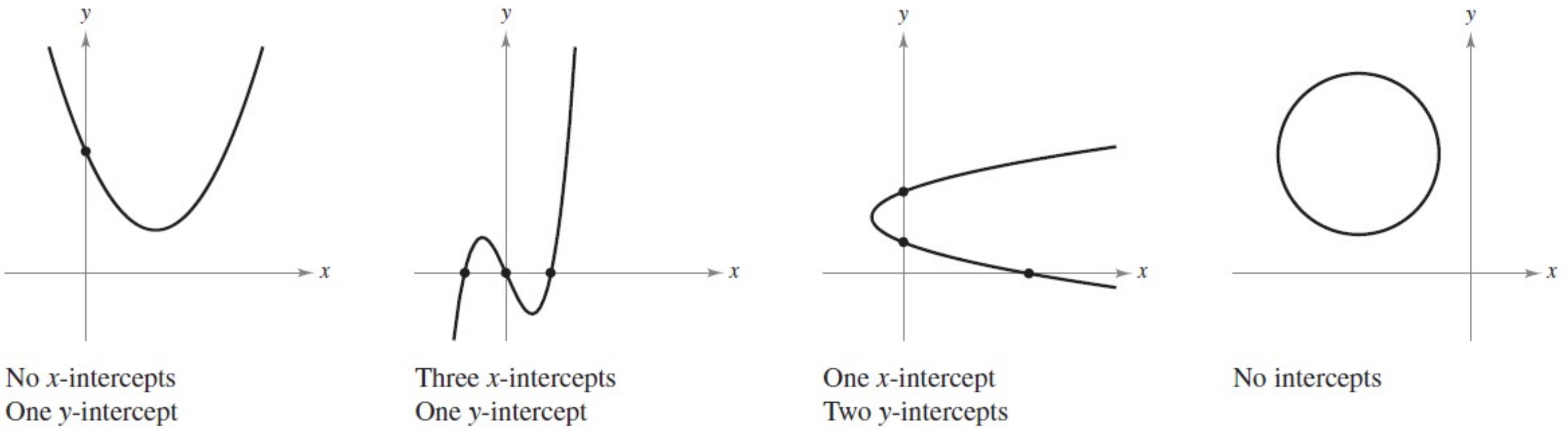


Figure P.5

Example 2 – *Finding x- and y-Intercepts*

Find the x-and y-intercepts of the graph of $y = x^3 - 4x$.

Solution:

To find the x-intercepts, let y be zero and solve for x.

$$x^3 - 4x = 0 \qquad \text{Let } y \text{ be zero.}$$

$$x(x - 2)(x + 2) = 0 \qquad \text{Factor.}$$

$$x = 0, 2, \text{ or } -2 \qquad \text{Solve for } x.$$

Example 2 – *Solution*

cont'd

Because this equation has three solutions, you can conclude that the graph has three x-intercepts:

(0, 0), (2, 0), and (–2, 0). x-intercepts

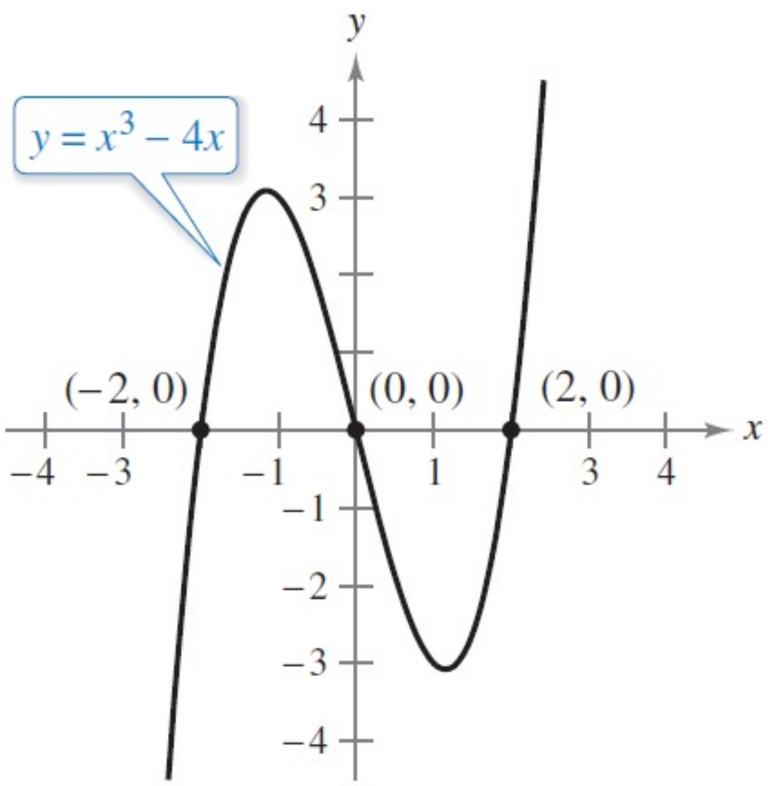
To find the y-intercepts, let x be zero. Doing this produces $y = 0$. So, the y-intercept is

(0, 0). y-intercept

Example 2 – *Solution*

cont'd

(See Figure P.6.)



Intercepts of a graph

Figure P.6

Symmetry of a Graph

Symmetry of a Graph

Knowing the symmetry of a graph before attempting to sketch it is useful because you need only half as many points to sketch the graph.

The following three types of symmetry can be used to help sketch the graphs of equations.

Symmetry of a Graph

1. A graph is **symmetric with respect to the *y*-axis** if, whenever (x, y) is a point on the graph, then $(–x, y)$ is also a point on the graph. This means that the portion of the graph to the left of the *y*-axis is a mirror image of the portion to the right of the *y*-axis.

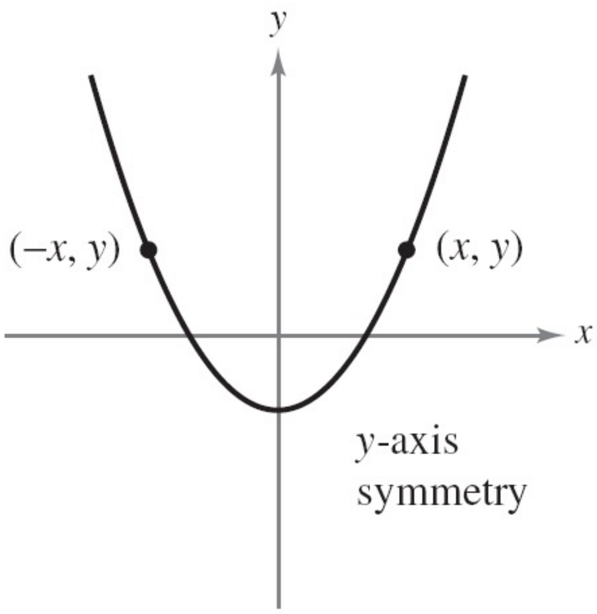


Figure P.7(a)

Symmetry of a Graph

2. A graph is **symmetric with respect to the *x*-axis** if, whenever (x, y) is a point on the graph, then $(x, -y)$ is also a point on the graph. This means that the portion of the graph above the *x*-axis is a mirror image of the portion below the *x*-axis.

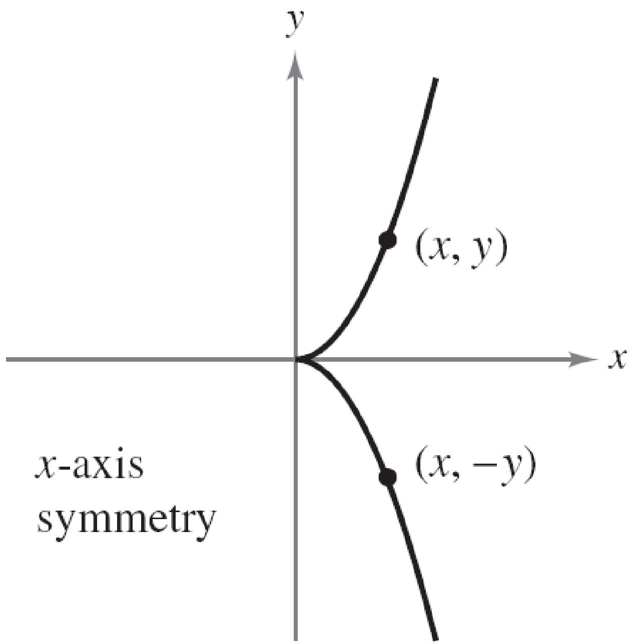


Figure P.7(b)

Symmetry of a Graph

3. A graph is **symmetric with respect to the origin** if, whenever (x, y) is a point on the graph, then ($-x$, $-y$) is also a point on the graph. This means that the graph is unchanged by a rotation of 180° about the origin.

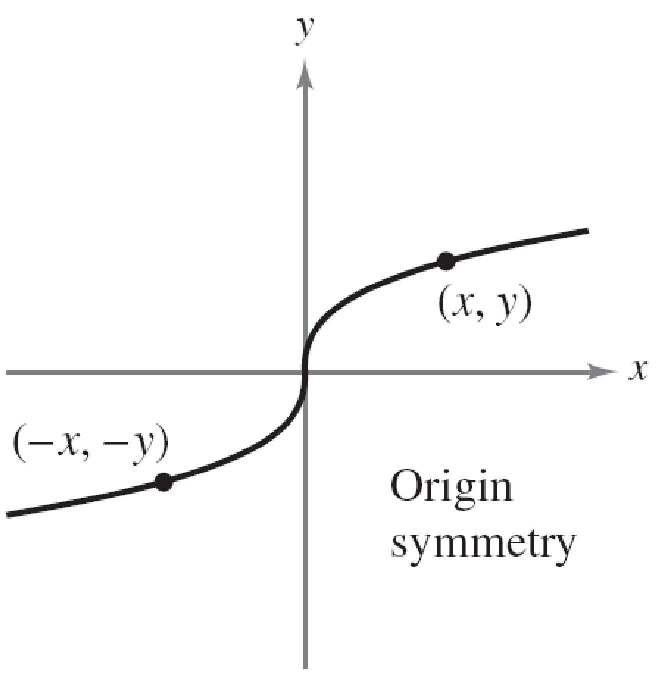


Figure P.7(c)

Symmetry of a Graph

Tests for Symmetry

1. The graph of an equation in x and y is symmetric with respect to the y-axis when replacing x by $-x$ yields an equivalent equation.
2. The graph of an equation in x and y is symmetric with respect to the x-axis when replacing y by $-y$ yields an equivalent equation.
3. The graph of an equation in x and y is symmetric with respect to the origin when replacing x by $-x$ and y by $-y$ yields an equivalent equation.

Symmetry of a Graph

The graph of a polynomial has symmetry with respect to the y-axis when each term has an even exponent (or is a constant). For instance, the graph of

$$y = 2x^4 - x^2 + 2$$

has symmetry with respect to the y-axis. Similarly, the graph of a polynomial has symmetry with respect to the origin when each term has an odd exponent.

Example 3 – *Testing for Symmetry*

Test the graph of $y = 2x^3 - x$ for symmetry with respect to (a) the y-axis and (b) the origin.

Solution:

(a) $y = 2x^3 - x$ Write original equation.

$y = 2(-x)^3 - (-x)$ Replace x by $-x$.

$y = -2x^3 + x$ Simplify. This result is *not* an equivalent equation.

Because replacing x by $-x$ does *not* yield an equivalent equation, you can conclude that the graph of $y = 2x^3 - x$ is *not* symmetric with respect to the y-axis.

Example 3 – *Solution*

cont'd

(b) $y = 2x^3 - x$ Write original equation.

$-y = 2(-x)^3 - (-x)$ Replace *x* by –*x* and *y* by –*y*.

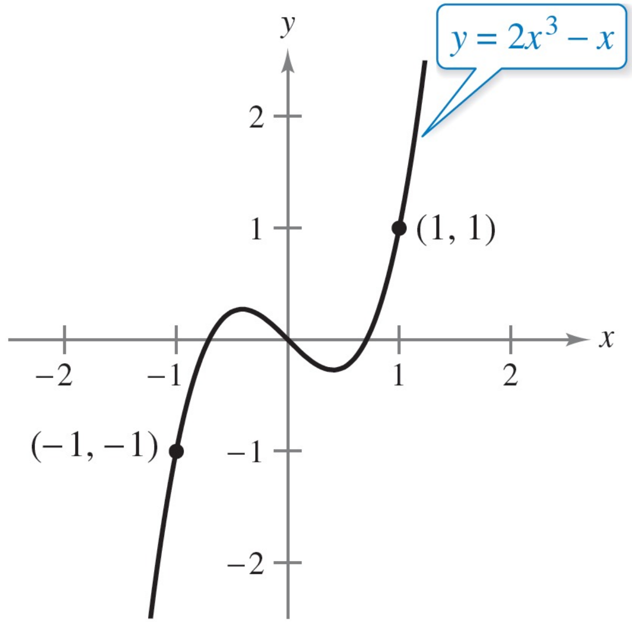
$-y = -2x^3 + x$ Simplify.

$y = 2x^3 - x$ Equivalent equation.

Example 3 – *Solution*

cont'd

Because replacing x by $-x$ and y by $-y$ yields an equivalent equation, you can conclude that the graph of $y = 2x^3 - x$ is symmetric with respect to the origin, as shown in Figure P.8.



Origin symmetry

Figure P.8

Points of Intersection

Points of Intersection

A **point of intersection** of the graphs of two equations is a point that satisfies both equations.

You can find the point(s) of intersection of two graphs by solving their equations simultaneously.

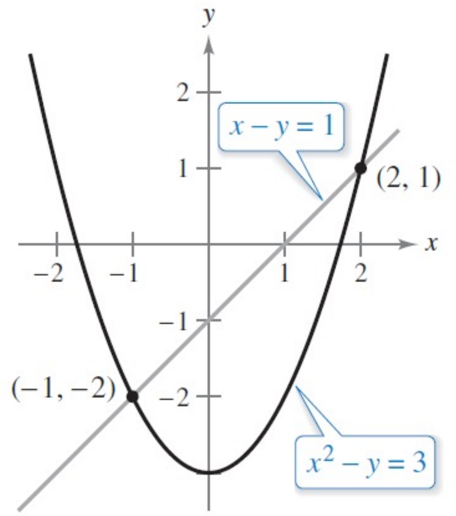
Example 5 – *Finding Points of Intersection*

Find all points of intersection of the graphs of

$$x^2 - y = 3 \quad \text{and} \quad x - y = 1.$$

Solution:

Begin by sketching the graphs of both equations in the *same* rectangular coordinate system, as shown in Figure P.10.



Two points of intersection

Figure P.10

Example 5 – *Solution* cont'd

From the figure, it appears that the graphs have two points of intersection. You can find these two points as follows.

$y = x^2 - 3$	Solve first equation for *y.*
$y = x - 1$	Solve second equation for y.
$x^2 - 3 = x - 1$	Equate *y*-values.
$x^2 - x - 2 = 0$	Write in general form.
$(x - 2)(x + 1) = 0$	Factor.
$x = 2 \text{ or } -1$	Solve for *x.*

Example 5 – *Solution*

cont'd

The corresponding values of y are obtained by substituting $x = 2$ and $x = -1$ into either of the original equations.

Doing this produces two points of intersection:

$(2, 1)$ and $(-1, -2)$. Points of intersection

Mathematical Models

Mathematical Models

Real-life applications of mathematics often use equations as **mathematical models**.

In developing a mathematical model to represent actual data, you should strive for two (often conflicting) goals: accuracy and simplicity.

That is, you want the model to be simple enough to be workable, yet accurate enough to produce meaningful results.

Example 6 – *Comparing Two Mathematical Models*

The Mauna Loa Observatory in Hawaii records the carbon dioxide concentration *y* (in parts per million) in Earth's atmosphere. The January readings for various years are shown in Figure P.11.

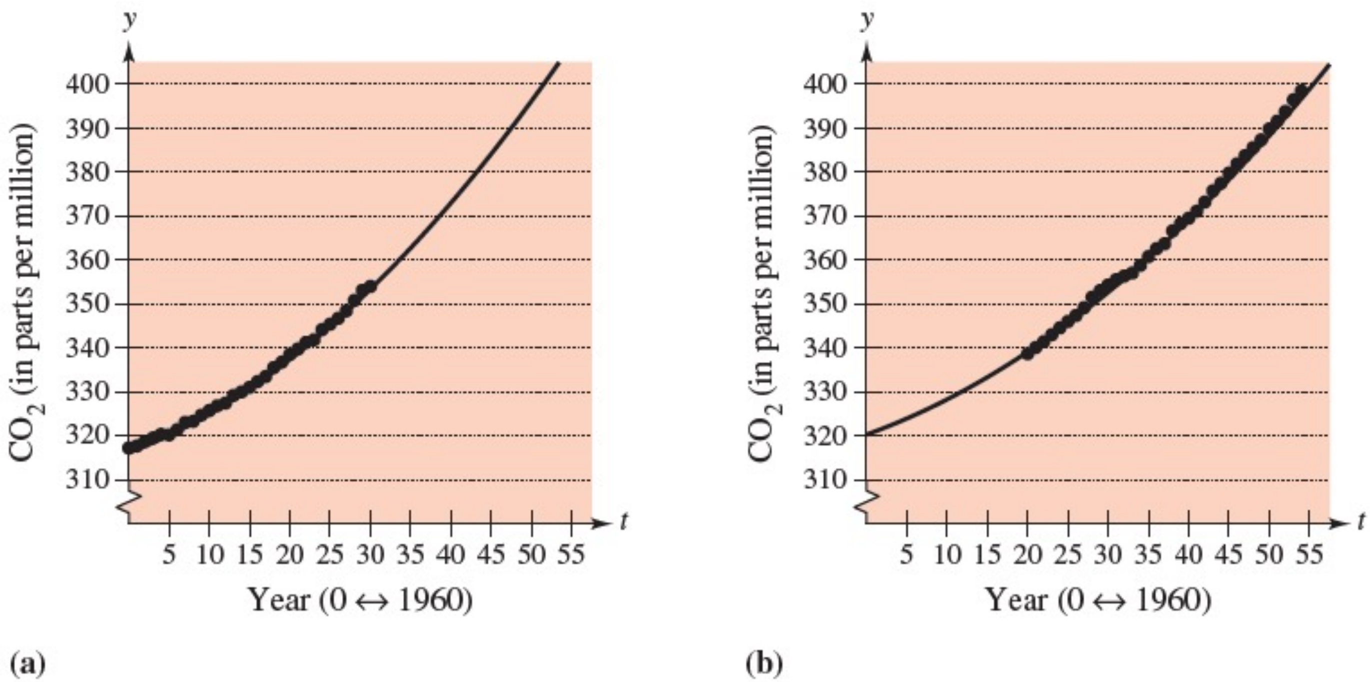


Figure P.11

Example 6 – *Comparing Two Mathematical Models* cont'd

In the July 1990 issue of *Scientific American*, these data were used to predict the carbon dioxide level in Earth's atmosphere in the year 2035, using the quadratic model

$$y = 0.018t^2 + 0.70t + 316.2 \qquad \text{Quadratic model for 1960–1990 data}$$

where $t = 0$ represents 1960, as shown in Figure P.11(a).

Example 6 – *Comparing Two Mathematical Models* cont’d

The data shown in Figure P.11(b) represent the years 1980 through 2014 and can be modeled by

$$y = 0.014t^2 + 0.66t + 320.3$$ Quadratic model for 1980–2014 data

where $t = 0$ represents 1960.

What was the prediction given in the *Scientific American* article in 1990? Given the second model for 1980 through 2014, does this prediction for the year 2035 seem accurate?

Example 6 – *Solution*

To answer the first question, substitute t = 75 (for 2035) into the first model.

$$y = 0.018(75)^2 + 0.70(75) + 316.2 = 469.95 \qquad \text{Model for 1960–1990 data}$$

So, the prediction in the *Scientific American* article was that the carbon dioxide concentration in Earth's atmosphere would reach about 470 parts per million in the year 2035.

Example 6 – *Solution*

cont'd

Using the model for the 1980–2014 data, the prediction for the year 2035 is

$$y = 0.014(75)^2 + 0.66(75) + 320.3 = 448.55. \qquad \text{Model for 1980–2014 data}$$

So, based on the model for 1980–2014, it appears that the 1990 prediction was too high.